TEMPERATURE DISTRIBUTION IN LAMINAR FLOW OF AN INCOMPRESSIBLE FLUID, FLOWING IN A RECTANGULAR CHANNEL WITH BOUNDARY CONDITIONS OF THE SECOND KIND

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UDC 536.22:532.501.33

The distribution of temperature across the channel has been obtained for laminar flow of a fluid in an infinite rectangular channel, in the case when the heat flux to the walls is a function of the coordinates.

For the case of a fluid in a rectangular channel, the differential equations of motion and energy can be written, respectively, in the following form:

$$\frac{\partial^2 W_z}{\partial x^2}+\frac{\partial^2 W_z}{\partial y^2}=\frac{1}{\mu}\,\frac{\partial P}{\partial z}; \tag{1}$$

$$\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}=\frac{W_z}{\varkappa}\,\frac{\partial t}{\partial z}. \tag{2}$$

Equations (1) and (2) have been obtained with the following assumptions:

1) Stationary flow of an incompressible fluid with constant physical properties, independent of temperature, is considered downstream of the section where the heat flux and the temperature became stabilized.

2) The fluid flow is laminar, the velocity components $W_x = W_y = 0$, and P = const across the channel.

3) No account is taken of the effect of mass forces, the generation of frictional heating, and heat conduction of the walls.

In this formulation, the solution was given in [1] for the temperature distribution in a rectangular channel, for the case of constant heat flux to the wall. Temperature distributions were obtained, both for the case of heating from all sides, and for heating from two opposite sides. Siegel and Savino [6] considered the problem of temperature distribution in a rectangular channel with constant heat flux to the walls, allowing for heat conduction of the walls.

In the present paper the temperature distribution across the channel section has been obtained when the heat flux to the walls is a function of the coordinates x or y, but does not vary along the stream, with z. For simplicity, the case considered is that of symmetrical heating, i.e., the heat flux to opposite walls of the channel is given as the same function of the coordinates.

The boundary conditions for this problem can be written in the form (Fig. 1):

$$\begin{aligned} y=0 \quad &\left(\frac{\partial t}{\partial y}\right)_0=-\varphi_1(x); \qquad W_z=0;\\ y=b \quad &\left(\frac{\partial t}{\partial y}\right)_b=\varphi_1(x); \qquad W_z=0; \end{aligned} \tag{3}$$

Power Engineering Institute, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 16, No. 1, pp. 30-37, January, 1969. Original article submitted March 15, 1968.

$$x = 0 \quad \left(\frac{\partial t}{\partial x}\right)_0 = -\psi_1(y); \qquad W_z = 0;$$
$$x = a \quad \left(\frac{\partial t}{\partial x}\right)_a = \psi_1(y); \qquad W_z = 0. \tag{4}$$

By converting in Eqs. (1)-(4) to the new variables:

$$X = \frac{x}{a}\pi, \; Y = \frac{y}{b}\pi,$$

we obtain:

$$\frac{\partial^2 W_z}{\partial X^2} + \frac{a^2}{b^2}\frac{\partial^2 W_z}{\partial Y^2} = \frac{a^2}{\pi^2}\frac{1}{\mu}\frac{\partial P}{\partial z} = A; \tag{5}$$

$$\frac{\partial^2 t}{\partial X^2} + \frac{a^2}{b^2}\frac{\partial^2 t}{\partial Y^2} = \frac{a^2}{\pi^2}\frac{W_z}{\varkappa}\frac{\partial t}{\partial z}; \tag{6}$$

$$Y = 0; \; \left(\frac{\partial t}{\partial Y}\right)_0 = -\varphi(X)\frac{b}{\pi}; \qquad W_z = 0;$$
$$Y = \pi; \; \left(\frac{\partial t}{\partial Y}\right)_\pi = \varphi(X)\frac{b}{\pi}; \qquad W_z = 0; \tag{7}$$

$$X = 0; \; \left(\frac{\partial t}{\partial X}\right)_0 = -\psi(Y)\frac{a}{\pi}; \qquad W_z = 0;$$
$$X = \pi; \; \left(\frac{\partial t}{\partial X}\right)_\pi = \psi(Y)\frac{a}{\pi}; \qquad W_z = 0. \tag{8}$$

In accordance with assumptions (1), Eqs. (5) and (6) can be solved independently. The solution of Eq. (5), satisfying conditions (7) and (8), has been given in [5], and has the following form:

$$W_z = \frac{4A}{\pi}\sum_{p=1,3,5,\ldots}^{\infty}\left[\frac{\operatorname{ch}\left(pY\frac{b}{a} - p\pi\frac{b}{2a}\right)}{\operatorname{ch}\frac{p\pi b}{2a}} - 1\right]\frac{\sin pX}{p^3}. \tag{9}$$

Since, in conditions of stabilized flow with constant physical properties, the local fluid temperature varies linearly along the channel [2], i.e., $\partial t/\partial z$ = const, Eq. (6) can be written in the following form:

$$\frac{\partial^2 t}{\partial X^2} + \frac{a^2}{b^2}\frac{\partial^2 t}{\partial Y^2} = B\sum_{p=1,3,5,\ldots}^{\infty}\left[\frac{\operatorname{ch}\left(pY\frac{b}{a} - p\pi\frac{b}{2a}\right)}{\operatorname{ch}\frac{p\pi b}{2a}} - 1\right]\frac{\sin pX}{p^3}, \tag{10}$$

where

$$B = \frac{4A}{\pi^3}\frac{a^2}{\varkappa}\frac{\partial t}{\partial z} = \text{const}.$$

To solve Eq. (10), with boundary conditions (7) and (8), we use a finite integral Fourier cosine transformation with respect to X, considered in detail in [3]. By multiplying both sides of Eq. (10) by $\cos kX dX$, and integrating from 0 to π, we obtain:

1) $k \neq 0$, k even

$$B\sum_{p=1,3,5,\ldots}^{\infty}\left[\frac{\operatorname{ch}\left(pY\frac{b}{a} - p\pi\frac{b}{2a}\right)}{\operatorname{ch}\frac{p\pi b}{2a}} - 1\right]\frac{1}{p^2}\frac{2}{(p^2 - k^2)} = \frac{a^2}{b^2}\frac{\partial^2 T}{\partial Y^2} - k^2 T + \frac{2a}{\pi}\psi(Y); \tag{11}$$

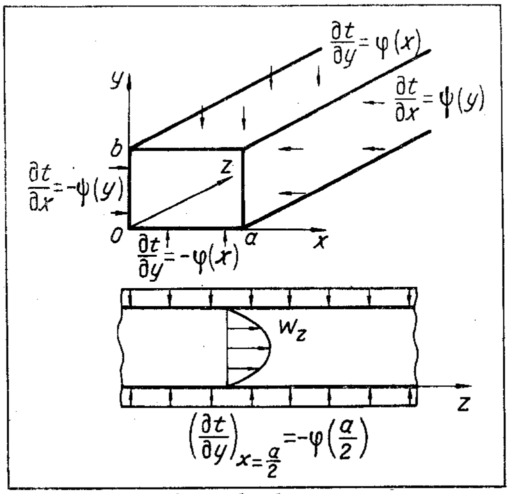

Fig. 1. The calculation scheme.

2) k = 0

$$B\sum_{p=1,3,5,\ldots}^{\infty}\left[\frac{\operatorname{ch}\left(pY\frac{b}{a}-p\pi\frac{b}{2a}\right)}{\operatorname{ch}p\pi\frac{b}{2a}}-1\right]\frac{2}{p^4}=\frac{a^2}{b^2}\frac{\partial^2 T_0}{\partial Y^2}+\frac{2a}{\pi}\psi(Y); \tag{12}$$

3) k ≠ 0, k odd

$$\frac{a^2}{b^2}\frac{\partial^2 T}{\partial Y^2}-k^2T+\frac{2a}{\pi}\psi(Y)=0. \tag{13}$$

The boundary conditions can be written in the following form:

k ≠ 0

$$Y=0;\quad \left(\frac{\partial T}{\partial Y}\right)_0=-\int_0^{\pi}\varphi(X)\frac{b}{\pi}\cos kXdX=-\lambda(k);$$
$$Y=\pi;\quad \left(\frac{\partial T}{\partial Y}\right)_{\pi}=\int_0^{\pi}\varphi(X)\frac{b}{\pi}\cos kXdX=\lambda(k); \tag{14}$$

k = 0

$$Y=0;\quad \left(\frac{\partial T_0}{\partial Y}\right)_0=-\int_0^{\pi}\varphi(X)\frac{b}{\pi}dX=-\lambda(0);$$
$$Y=\pi;\quad \left(\frac{\partial T_0}{\partial Y}\right)_{\pi}=\int_0^{\pi}\varphi(X)\frac{b}{\pi}dX=\lambda(0). \tag{15}$$

In Eqs. (11)-(15) it is assumed that

$$T=\int_0^{\pi}t\cos kXdX;\quad T_0=\int_0^{\pi}tdX.$$

Solving the differential equations (11)-(13), we obtain [4]:

1) k ≠ 0, k even

$$T=C_1\exp\left(k\frac{b}{a}Y\right)+C_2\exp\left(-k\frac{b}{a}Y\right)+F(Y), \tag{16}$$

where

$$F(Y) = B \sum_{p=1,3,\ldots}^{\infty} \left[\frac{\operatorname{ch}\left(pY\frac{b}{a} - p\pi\frac{b}{2a}\right)}{\operatorname{ch}\frac{p\pi b}{2a}(p^2 - k^2)} + \frac{1}{k^2} \right] \frac{1}{p^2} \frac{2}{(p^2 - k^2)} + f(Y); \quad (17)$$

f(Y) is the particular solution of the equation

$$\frac{a^2}{b^2} \frac{\partial^2 f(Y)}{\partial Y^2} - k^2 f(Y) = -\frac{2a}{\pi} \psi(Y); \quad (18)$$

2) $k \neq 0$, k odd

$$T = C_3 \exp\left(k\frac{b}{a}Y\right) + C_4 \exp\left(-k\frac{b}{a}Y\right) + f(Y); \quad (19)$$

3) $k = 0$

$$T_0 = C_{10} + C_{20}Y + F_0(Y); \quad (20)$$

$$F_0(Y) = B \sum_{p=1,3,\ldots}^{\infty} \frac{2}{p^4} \left[\frac{\operatorname{ch}\left(pY\frac{b}{a} - p\pi\frac{b}{2a}\right)}{p^2 \operatorname{ch}\frac{p\pi b}{2a}} - \frac{b^2}{a^2}\frac{Y^2}{2} \right] + f_0(Y), \quad (21)$$

and $f_0(Y)$ is the particular solution of the equation

$$\frac{a^2}{b^2} \frac{\partial^2 f_0(Y)}{\partial Y^2} = -\frac{2a}{\pi} \psi(Y). \quad (22)$$

The constants C_1, C_2, C_3, C_4, and C_{20} are determined from the boundary conditions (14) and (15). The constant C_{10} can be determined by assuming, for example, that $t = 0$ when $X = Y = 0$. The final expression for the temperature field across the channel can be written in the following form:

$$t = \frac{1}{\pi} T_0 + \frac{2}{\pi} \sum_{k=1}^{\infty} T \cos kX. \quad (23)$$

If, for example, a heat flux of constant value $q_0 = \text{const}$ is given on all the channel walls, the expression for the temperature field across the channel has the form

$$t = \frac{1}{\pi} \left\{ C_{10} + \left[\frac{q_0}{\lambda} \frac{b^2}{a} + B\frac{b^2}{a^2}\pi \sum_{p=1,3,\ldots}^{\infty} \frac{1}{p^4} \right] Y \right.$$

$$\left. + B \sum_{p=1,3,\ldots}^{\infty} \frac{2}{p^4} \left[\frac{\operatorname{ch}\left(pY\frac{b}{a} - \frac{p\pi b}{2a}\right)}{p^2 \operatorname{ch}\frac{p\pi b}{2a}} - \frac{b^2}{a^2}\frac{Y^2}{2} \right] - \frac{q_0 b^2}{\lambda a}\frac{Y^2}{\pi} \right\}$$

$$+ \frac{2}{\pi} \sum_{k=2,4,6,\ldots}^{\infty} \left[C_1 \exp\left(k\frac{b}{a}Y\right) + C_2 \exp\left(-k\frac{b}{a}Y\right) \right.$$

$$\left. + B \sum_{p=1,3,\ldots}^{\infty} \left(\frac{\operatorname{ch}\left(pY\frac{b}{a} - \frac{p\pi b}{2a}\right)}{\operatorname{ch}\frac{p\pi b}{2a}(p^2 - k^2)} + \frac{1}{k^2} \right) \frac{1}{p^2} \frac{2}{(p^2 - k^2)} + \frac{2a}{\pi k^2}\frac{q_0}{\lambda} \right] \cos kX, \quad (24)$$

where

$$B = \frac{q_0}{\lambda} \frac{a(a+b)}{b} \frac{1}{\sum_{p=1,3,\ldots}^{\infty} \left[\frac{2\operatorname{th}\frac{p\pi b}{2a}}{\frac{pb}{a}} - \pi \right] \frac{1}{p^4}};$$

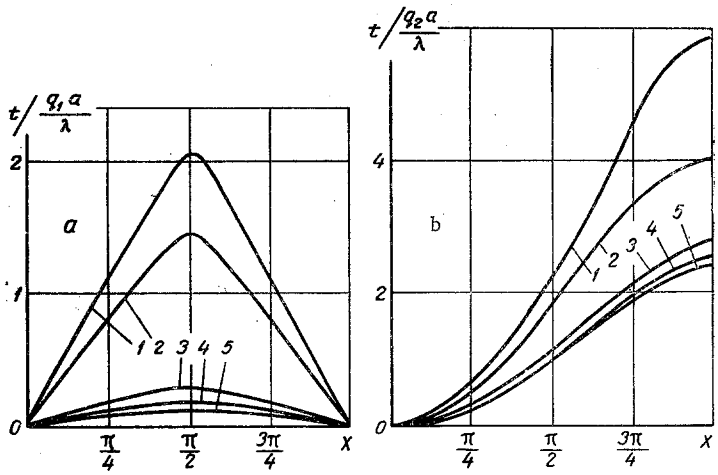

Fig. 2. Temperature distribution at the wall (Y = 0) for a sinusoidal (a) and a parabolic (b) distribution of heat flux:

a) $\left(\frac{\partial t}{\partial y}\right)_0 = -\left(\frac{\partial t}{\partial y}\right)_b = -\frac{q_1}{\lambda}\sin\frac{\pi x}{a}$; $\left(\frac{\partial t}{\partial x}\right)_0 = \left(\frac{\partial t}{\partial x}\right)_a = 0$;

1—a/b=7,0; 2—5,0; 3—2,0; 4—1,0; 5—0,2;

b) $\left(\frac{\partial t}{\partial y}\right)_0 = -\left(\frac{\partial t}{\partial y}\right)_b = -\frac{q_2}{\lambda}\,\frac{x^2\pi^2}{a^2}$; $\left(\frac{\partial t}{\partial x}\right)_0 = \left(\frac{\partial t}{\partial x}\right)_a = 0$;

1—a/b=3,3; 2—2,0; 3—1,0; 4—0,2; 5—0,1

$$C_1 = \frac{1+\exp\left(-k\frac{b}{a}\pi\right)}{k\left(\exp\left(-k\pi\frac{b}{a}\right)-\exp\left(k\pi\frac{b}{a}\right)\right)}\,2B\sum_{p=1,3,\ldots}^{\infty}\frac{\operatorname{th}\frac{p\pi b}{2a}}{p\,(p^2-k^2)^2};$$
$$C_2 = \frac{1+\exp\left(k\pi\frac{b}{a}\right)}{k\left(\exp\left(-k\pi\frac{b}{a}\right)-\exp\left(k\pi\frac{b}{a}\right)\right)}\,2B\sum_{p=1,3,\ldots}^{\infty}\frac{\operatorname{th} p\pi\frac{b}{2a}}{p\,(p^2-k^2)^2}, \tag{25}$$

and the value of C_{10} obtained from the condition t = 0 for X = Y = 0 is equal to

$$C_{10} = -2\sum_{k=2,4,\ldots}^{\infty}\left[C_1 + C_2 + B\sum_{p=1,3,\ldots}^{\infty}\frac{2}{k^2(p^2-k^2)^2} + \frac{2a}{\pi k^2}\,\frac{q_0}{\lambda}\right] - 2B\sum_{p=1,3,\ldots}^{\infty}\frac{1}{p^6}.$$

In the case when the heat flux to two opposite walls is a sinusoidally varying quantity, i.e.,

$$\left(\frac{\partial t}{\partial y}\right)_0 = -\left(\frac{\partial t}{\partial y}\right)_b = -\frac{q_1}{\lambda}\sin\frac{\pi x}{a},$$
$$\left(\frac{\partial t}{\partial x}\right)_0 = -\left(\frac{\partial t}{\partial x}\right)_a = 0,$$

we can obtain the following expression for the temperature field across the channel:

$$t = \frac{1}{\pi}\left\{C'_{10} + B'\frac{b^2}{a^2}\pi\sum_{p=1,3,\ldots}^{\infty}\frac{1}{p^4}Y + B'\sum_{p=1,3,\ldots}^{\infty}\frac{2}{p^4}\left[\frac{\operatorname{ch}\left(pY\frac{b}{a} - p\pi\frac{b}{2a}\right)}{p^2\operatorname{ch}\frac{p\pi b}{2a}}\right.\right.$$
$$\left.\left.-\frac{b^2}{a^2}\,\frac{Y^2}{2}\right]\right\} + \frac{2}{\pi}\sum_{k=2,4,\ldots}^{\infty}\left[C'_1\exp\left(k\frac{b}{a}Y\right)\right.$$

$$+C_2'\exp\left(-k\frac{b}{a}Y\right)\Bigg]+B'\sum_{p=1,3,\ldots}^{\infty}\left[\left(\frac{\mathrm{ch}\left(pY\frac{b}{a}-p\pi\frac{b}{2a}\right)}{\mathrm{ch}\frac{p\pi b}{2a}(p^2-k^2)}+\frac{1}{k^2}\right)\frac{1}{p^2}\frac{2}{(p^2-k^2)}\right]\cos kX, \tag{26}$$

where

$$B'=\frac{q_1}{\lambda}\frac{a^2}{b}\frac{2}{\pi}\frac{1}{\sum\limits_{p=1,3,\ldots}^{\infty}\left[\frac{2\mathrm{th}\frac{p\pi b}{2a}}{\frac{pb}{a}}-\pi\right]\frac{1}{p^4}},$$

and the constants C_{10}', C_1', C_2' can easily be determined from the boundary conditions.

Figure 2a shows curves of temperature at the wall ($Y = 0$), obtained from Eq. (26) with a sinusoidal heat flux distribution on two opposite walls, for channels with various ratios of sides, and Fig. 2b gives curves of temperature distribution at the wall ($Y = 0$) for the case when the heat flux is parabolic on two opposite walls,

$$q=q_2X^2=q_2\frac{x^2\pi^2}{a^2},$$

as a function of the ratio of sides ($t_{X=Y=0} = 0$).

Since the expressions for the temperature and velocity fields are infinite series, the first few terms are used to calculate according to the corresponding formulas, depending on the convergence of the respective series. For example, using Eq. (9), it is enough to take only the first two terms of the series ($p = 1$, $p = 3$), since the coefficient of the third term ($p = 5$) is less than 0.01 of the coefficient of the first term of the series.

NOTATION

W_z	is the stream velocity;
x, y	are the coordinates perpendicular to the stream direction;
z	is the coordinate along the stream;
λ	is the thermal conductivity of the fluid;
$\varkappa$	is the thermal diffusivity of the fluid;
μ	is the viscosity of the fluid;
P	is the pressure;
t	is the temperature;
a, b	are the channel width and height;
q_0, q_1, q_2	are the specific heat fluxes to the wall.

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